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ON THE CONE OF SECOND ORDER WHICH IS ANALOGOUS TO THE NINE-POINT CONIC.

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The theorem on the existence of the so-called nine-point conic may be stated in its most general form as follows :—

Let $ABCD$ be any complete quadrangle whose six sides, AB, AC, AD, BC, BD, CD are cut by an arbitrary straight line a in the points P, Q, R, S, T, V, respectively ; and let E, F, H, K, L, M be the harmonic conjugates of these points with respect to the pairs of vertices of the quadrangle, so that $AEBP$, $AFCQ$, etc., are harmonic ranges. Then a conic may be passed through the six points E, F, H, K, L, M, on which will also lie the three points of intersection, X, Y, Z, of the pairs of opposite sides of the quadrangle.

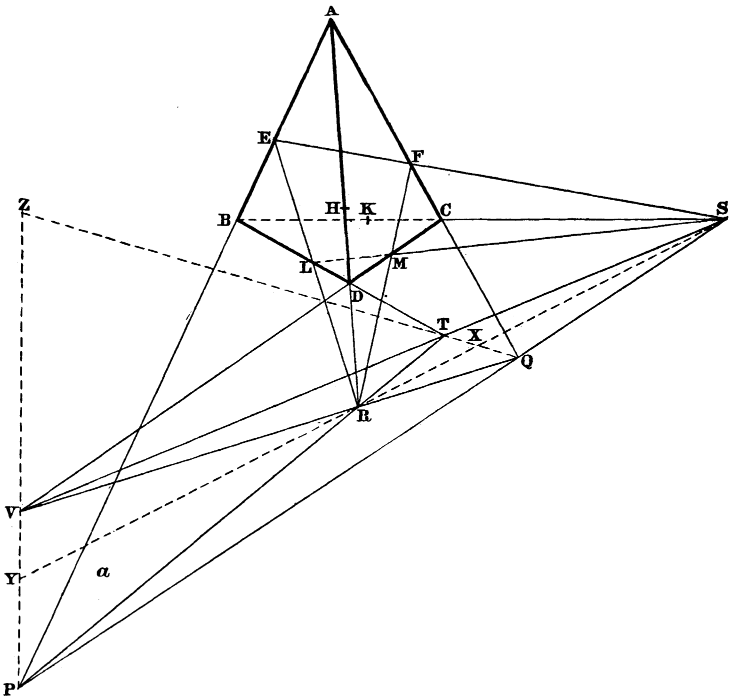
In this form the theorem admits of a very simple geometrical demonstration, and embraces as special cases the ordinary nine-point conic and nine-point circle. For, if the line a be the infinitely distant line in the plane, the harmonic conjugate points become the bisection points of the sides of the quadrangle, and if at the same time the quadrangle be such that the three pairs of opposite sides intersect at right angles, the conic becomes a circle.

Since the line a may have any position in the plane, it determines the doubly infinite system of conics through the three points X, Y, Z. If it pass through one of these points, for example through X, the corresponding conic degenerates into two straight lines of which the one passes through X, and the other is the line YZ. If the line a pass through two of these points, say X and Y, then the corresponding conic breaks up into the two straight lines XZ and YZ.

We readily obtain an analogous relation in space of three dimensions by a study of the following configuration :—

Suppose $ABCD$ be any tetrahedron whose faces are cut by an arbitrary plane a, not passing through any of the vertices, and without loss of generality we may suppose a to cut all the faces extended. Let a cut the edges AB, AC, AD, BC, BD, CD, in P, Q, R, S, T, V, respectively, and to each of these points find the harmonic conjugate with respect to the pair of tetrahedral points lying on the same edge. Let these conjugate points be E, F, H, K, L, M, so that $AEBP$, $AFCQ$, etc. are harmonic ranges along the edges of the tetrahedron. Then the following theorem holds true :—

From any point O in the plane a, these six points, E, F, H, K, L, M, may be projected by a cone of the second order, which however breaks up into two



planes for certain special positions of the vertex; and, if from O be drawn the three rays meeting pairs of opposite edges of the tetrahedron, these will also lie on this cone.

1. The points P, Q, R, S, T, V lie by threes in straight lines, the intersections of the faces of the tetrahedron with a. Moreover, S, F, and E are collinear. For, since $S(AEBP)$ is a harmonic pencil lying in the plane ABC, and is cut by the straight line $AFCQ$ so that C and Q lie respectively in the rays SB and SP, then must F fall in the ray SE since by supposition $AFCQ$ is a harmonic range. So also S, M, and L are collinear, as are all sets of similarly situated points.

Consider E, F, H, M, L, K to be a gauche hexagon with EF and ML, FH and LK, HM and KE, as pairs of opposite sides. Let O be any point in space, and from it project the hexagon.

The planes OEF and OML will intersect in the line OS,
" OFH " OLK " " OV,
" OHM " OKE " " OQ.

By the space extension of Pascal's Theorem, a cone of the second order whose vertex is O will circumscribe the hexagon when and only when OS, OV, and

OQ lie in one plane. These lines are coplanar if O lies in the plane SVQ, i. e. in the plane α.

2. The points group themselves by fours in planes, as follows:

$EFLM$ in a plane cutting α in the line RS,
$EHKM$ " " α " " TQ,
$FHLK$ " " α " " PV,

Hence, out of any point in one of the lines RS, TQ, or PV, the six points are projected by two planes, one of which contains four of the points, the other, the remaining two points. It is also easily seen that the six points are projected by two planes, three points in each plane, out of any point in the four lines of intersection of the faces of the tetrahedron with α, so that when the vertex is a point in any one of these seven lines, which are the four sides and three diagonals of a complete quadrilateral, the cone of projection breaks up into two planes.

3. Let OG be the ray drawn from the vertex of the cone to meet any pair of opposite edges of the tetrahedron, for example, AB in G_1 and CD in G_2.

Then the planes OG_1E and OFK intersect in the line OP,
OEH " OKM " " OT,
OHF " OMG_2 " " OV.

But since O lies in the plane PTV, or α, the lines OP, OT, OV are coplanar, and hence the ray OG, and likewise each of the other two rays similarly drawn, lies on the cone of second order determined by the five rays $O(E, H, F, K, M)$. But this is the same cone on which we have shown the ray OL to lie. Hence the theorem.

Since E, F, L, M lie in one plane, EM and FL, lines joining the points in opposite edges of the tetrahedron, must intersect. Similarly, EM and HK, HK and FL intersect. As these lines EM, HK, and FL are not coplanar and each intersects the other two, they must pass through one point. As an interesting special case let α be parallel to one of the faces of the tetrahedron, for example, BCD. Then K, L, M become the middle points of the edges in this face, while E, F, H lie in a plane parallel to this face. Therefore,

If the three edges of a tetrahedron which lie in one face be bisected and the remaining three edges be cut by a plane parallel to this face, then the three lines joining the points so marked out in the pairs of opposite edges intersect in one point, and conversely.

If the plane α move out to infinity we have the well known theorem that " the joined lines of the middle points of the three pairs of opposite edges in any tetrahedron meet in a point." In this case also we observe that *the six lines which are drawn parallel to one another, in any direction, through the middle points of the edges of any tetrahedron and the three lines parallel to these* *which meet the pairs of opposite edges of this tetrahedron, all lie on a cylinder of the second order.*

In the general case, the point of intersection, W, of the three joined lines of the points marked out in the pairs of opposite edges bears many interesting relations to the plane α with reference to the tetrahedron. To each plane α in space is correlated in this way but one point W, and conversely. In such a system of correlation, however, the faces and edges of the tetrahedron to which reference is made, are singular planes and lines, since to any face all the points in that face are correlated, and to a point in any edge, all the planes through that edge are correlated.

From what has already been shown it readily follows; that

If there be given any six points in space which lie in pairs on three straight lines through a point W; a plane α may be determined out of any point of which the six points are projected by a cone of the second order, viz., the plane through the harmonic conjugates of W with respect to the three pairs of points.

The plane α is correlated to the point W not only with reference to the tetrahedron $ABCD$, but also with reference to what may be called the anti-tetrahedron; viz., that whose faces are determined by the points EHF, FKM, MLH, EKL. The faces and edges of this tetrahedron cut α in the same lines and points as do those of the original tetrahedron. Moreover, the choice of this plane α, to which the point W is correlated, determines, at the same time, three other planes β, γ, δ, and three other points X, Y, Z, which bear the same relation to each other, respectively, as do α and W. The three points X, Y, Z lie in the plane α, and are those points in which the lines EM, FL, HK meet the plane. Thus we obtain a new tetrahedron $WXYZ$ which is correlated to $ABCD$ in such a manner that if from any one of its vertices, for example, from Y, lines be drawn to meet the three pairs of opposite edges of $ABCD$, the six points so marked out on these edges are projected from any point in the opposite face, WXZ, by a cone of second order, on which will also lie the rays drawn from this point of projection to meet the pairs of opposite edges of the original tetrahedron.

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